The stability of zonal flows in barotropic fluid on a rotating sphere(Mathematical Aspects on Nonlinear Waves)

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The stability of zonal flows in barotropic fluid on a rotating sphere*
(回転球面上の帯状流の安定性について)

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The stability of zonal flows in barotropic fluid on a rotating sphere is studied in linear and nonlinear cases. In the linear stability analysis, eigenvalues and eigenmodes are obtained numerically and compared with Rayleigh's classical stability theorem. Shepherd's bound on the finite-amplitude saturation of wave disturbances to unstable zonal flows is modified to obtain a more rigorous bound. The bound is found to give good estimates of the result of nonlinear time-integrations.

1. Introduction

The linear stability of zonal flows on a plane has been investigated extensively since Rayleigh(1880). Kuo(1949) obtained a similar integral theorem on a β-plane. Baines(1976) examined the stability on a rotating sphere by a numerical eigenvalue analysis, and pointed out that the critical intensity for zonal flows to be unstable is close to that obtained from Rayleigh's classical criterion.

If the zonal flow is unstable, disturbances grow and have finite amplitudes. There have been several studies to obtain the upper bounds on the growth of the instabilities by weakly nonlinear theory. However, they are restricted to instabilities that are only weakly supercritical. Recently, Shepherd(1988) showed a novel method to obtain rigorous, full-nonlinear upper bounds. The method relies on the nonlinear Lyapunov stability theorem on stable shear flows (Arnol'd 1966), which is reviewed in detail by McIntyre & Shepherd(1987).

In this paper, we examine the linear stability of zonal flows in barotropic fluid on a rotating sphere by eigenvalue analysis following Baines(1976). As for the nonlinear phase of the instability we modify Shepherd's bound and compare these bounds with the result of numerical time-integrations. The governing equation is reviewed in §2, linear stability analysis is presented in §3, and nonlinear analysis is in §4. Discussions are given in §5. A new integral theorem is derived for the linear stability of some zonal flows in Appendix A, and classical integral theorems in the spherical geometry are summarized in Appendix B.

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2. Basic equations

The system under consideration is barotropic flow on a rotating sphere, which is governed by the conservation law of the absolute vorticity $q \equiv \nabla^2\psi + 2\Omega\mu$:

$$\frac{Dq}{Dt} \equiv \frac{\partial q}{\partial t} + \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \lambda}\frac{\partial q}{\partial \mu} - \frac{\partial \psi}{\partial \mu}\frac{\partial q}{\partial \lambda}\right) = 0, \tag{2.1}$$

where $\psi(\lambda, \mu, t)$ is the streamfunction, λ the longitude, μ sine of the latitude, t the time, a the radius of the sphere, Ω the angular speed of rotation of the sphere, and ∇^2 the horizontal Laplacian:

$$\nabla^2 \equiv \frac{1}{a^2}\left[\frac{1}{1-\mu^2}\frac{\partial^2}{\partial \lambda^2} + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial}{\partial \mu}\right\}\right]. \tag{2.2}$$

To study (2.1) numerically, we express the dependent variable ψ in spectral form by expanding it in spherical harmonics:

$$\psi(\lambda, \mu, t) = a^2\Omega \sum_{n=0}^{N}\sum_{m=-n}^{n} \psi_n^m(t) P_n^m(\mu) e^{im\lambda}, \tag{2.3}$$

where N is the truncation number, and $P_n^m(\mu)$ is the Legendre function defined as follows:

$$P_n^m(\mu) \equiv \sqrt{(2n+1)\frac{(n-m)!}{(n+m)!}}\frac{(-1)^n}{2^n n!}(1-\mu^2)^{\frac{m}{2}}\frac{d^{n+m}}{d\mu^{n+m}}(1-\mu^2)^n, \tag{2.4}$$

$$P_n^{-m}(\mu) \equiv P_n^m(\mu). \qquad (0 \le m \le n) \tag{2.5}$$

Substituting (2.3) into (2.1), multiplying by $P_n^m(\mu)e^{-im\lambda}$ and integrating over the whole sphere, we obtain a spectral barotropic vorticity equation

$$\frac{d\psi_n^m}{dt} = -\frac{i\Omega}{n(n+1)}\left(2m\psi_n^m + \sum_{n_1=0}^{N}\sum_{n_2=0}^{N}\sum_{m_1=-n_1}^{n_1}\sum_{m_2=-n_2}^{n_2} C_{nn_1n_2}^{mm_1m_2}\psi_{n_1}^{m_1}\psi_{n_2}^{m_2}\right), \tag{2.6}$$

$$\begin{aligned} C_{nn_1n_2}^{mm_1m_2} &\equiv \tfrac{1}{4}\{n_1(n_1+1) - n_2(n_2+1)\}\,\delta_m^{m_1+m_2} \\ &\times \int_{-1}^{1} P_n^m\left(m_1 P_{n_1}^{m_1}\frac{dP_{n_2}^{m_2}}{d\mu} - m_2 P_{n_2}^{m_2}\frac{dP_{n_1}^{m_1}}{d\mu}\right)d\mu. \end{aligned} \tag{2.7}$$

In this paper, we consider the stability of a zonal flow of $\overline{\psi}(\mu)$ to infinitesimal disturbances $\psi'(\lambda, \mu, t)$. The appropriate linearized form of (2.1) for the disturbances is

$$\left(\frac{\partial}{\partial t} + \overline{\omega}\frac{\partial}{\partial \lambda}\right)\nabla^2\psi' + \frac{1}{a^2}\frac{d\overline{q}}{d\mu}\frac{\partial \psi'}{\partial \lambda} = 0, \tag{2.8}$$

where $\overline{q} \equiv \nabla^2\overline{\psi} + 2\Omega\mu$, and $\overline{\omega}$ is the angular speed of the zonal flow:

$$\overline{\omega}(\mu) \equiv -\frac{1}{a^2}\frac{d\overline{\psi}}{d\mu}. \tag{2.9}$$

The spectral form of (2.8) is

$$\frac{d\psi_n'^m}{dt} = -\frac{i\Omega}{n(n+1)}\left(2m\psi_n'^m + 2\sum_{n_1=0}^{N} D_{nn_1}^m \psi_{n_1}'^m\right), \tag{2.10}$$

where

$$D^m_{nn_1} \equiv \sum_{n_2=0}^{N} C^{mm0}_{nn_1n_2} \overline{\psi}^0_{n_2}, \tag{2.11}$$

$$\overline{\psi}(\mu) = a^2\Omega \sum_{n=0}^{N} \overline{\psi}^0_n P^0_n(\mu), \tag{2.12}$$

$$\psi'(\lambda, \mu, t) = a^2\Omega \sum_{n=0}^{N} \sum_{m=-n}^{n} \psi'^m_n(t) P^m_n(\mu) e^{im\lambda}. \tag{2.13}$$

Note that the linear system is divided into subsets for each zonal wavenumber m.

3. Linear stability

We consider the stability of a zonal flow the streamfunction of which is represented by a single Legendre polynomial:

$$\overline{\psi}(\mu) \equiv a^2\Omega\overline{\psi}^0_3 P^0_3(\mu). \tag{3.1}$$

This is the lowest unstable mode, because $\overline{\psi}^0_1$ and $\overline{\psi}^0_2$ modes are stable at any amplitude(see Baines 1976, also Appendix A). Figure 1 shows latitudinal profiles of $\overline{u}(\mu)$, $\overline{\omega}(\mu)$, and $\overline{q}(\mu)$ for $\overline{\psi}^0_3 = -0.1$. The zonal flow is positive (eastward) in middle and high latitudes, while it is negative (westward) in low latitudes. The angular speed increases monotonously from the equator to the poles. The absolute vorticity changes the sign of its gradient at two latitudes, and therefore the zonal flow may be unstable from Rayleigh's criterion on a sphere (Appendix B). The critical value is obtained for the zonal flow of $\overline{\psi}^0_3$: it may be unstable when $\overline{\psi}^0_3 < -\frac{1}{9\sqrt{7}} \approx -0.042$, or $\overline{\psi}^0_3 > \frac{1}{36\sqrt{7}} \approx 0.0105$.

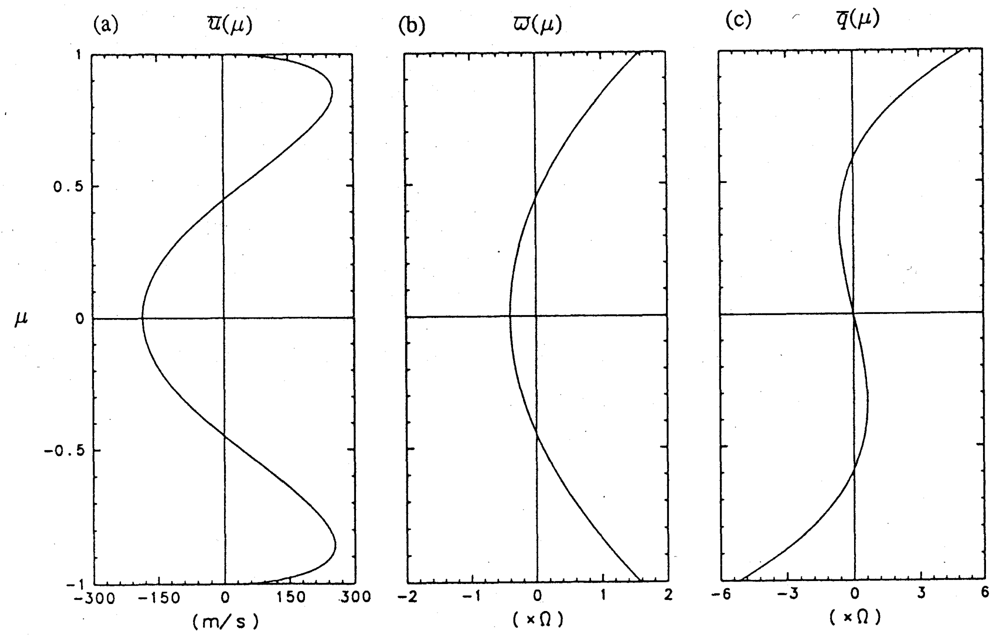

Figure 1. Latitudinal profiles of (a) $\overline{u}(\mu)$, (b) $\overline{\omega}(\mu)$, and (c) $\overline{q}(\mu)$ for $\overline{\psi}^0_3 = -0.1$.

To study the stability of a zonal flow numerically, we assume exponential time dependence $e^{-i\sigma t}$ in equation (2.10) and transform the equation into an algebraic eigenvalue equation. Eigenvalues σ and eigenmodes are obtained with QR method. Here, the angular speed of the eigenmode is σ_r/m and the growth rate is σ_i, where $\sigma_r \equiv Re(\sigma)$ and $\sigma_i \equiv Im(\sigma)$. If we define the complex angular speed as $c \equiv \sigma/m$, the angular speed is c_r and the growth rate mc_i.

As the zonal flow $\overline{\psi}_3^0$ is stable to wave disturbances with $m \geq 3$ (see the proof in Appendix A), the eigenvalue problem is solved for $m = 1$ and 2. The growth rates and the angular speed for $-1 \leq \overline{\psi}_3^0 \leq 1$ are shown in figure 2 ($N = 50$). The critical value of $\overline{\psi}_3^0$ where the zonal flow becomes unstable is -0.047 (to a disturbance $m = 2$), and 0.015 (to $m = 1$). Rayleigh's criterion gives a good estimate of the critical value, as Baines(1976) pointed out already. A pair of stable and unstable modes bifurcates ($\sigma = \sigma_r \pm i\sigma_i$) at the critical point. Minute structure of σ_i is observed in figure 2(c) for $m = 2$ around $\overline{\psi}_3^0 \approx 0.03$.

When $\overline{\psi}_3^0 = 0$, the angular speed of the eigenmodes corresponds to that of Rossby-Haurwitz waves $c_r = -\dfrac{2\Omega}{n(n+1)}(n = 1, 2, \ldots, N)$. On the other hand, most of the eigenmodes have singularity and their angular speed lies between the maximum and the minimum of the zonal angular speed $\overline{\omega}(\mu)$ when $\overline{\psi}_3^0 \neq 0$. The angular speed of the unstable modes is continuously connected to that of a Rossby-Haurwitz wave with ($m = 1, n = 2$) and with ($m = 2, n = 2$). There are some exceptional eigenmodes the angular speed of which is independent of the amplitude of $\overline{\psi}_3^0$; the horizontal lines of $c_r = -\Omega$ and $-\Omega/6$ in figure 2(b), and $c_r = -\Omega/6$ in figure 2(d). They are Rossby-Haurwitz waves with ($m = 1, n = 1$), ($m = 1, n = 3$) and ($m = 2, n = 3$) which do not interact with the zonal flow.

Figure 3 shows an example of the eigenmodes for $m = 2$ at $\overline{\psi}_3^0 = -0.1$, which consist of 25 symmetric modes with respect to the equator ($\mu = 0$) and 24 antisymmetric modes for the present truncation $N = 50$. The unstable mode (a) has a well-known latitudinal structure, tilting of trough and ridge lines. This structure indicates the latitudinal vorticity flux the convergence of which acts to reduce negative gradient of the absolute vorticity $\overline{q}_\mu$ shown in figure 1(c). On the other hand, the stable mode (b) has the opposite structure. The neutral modes are divided into the symmetric modes (c) and the antisymmetric modes (d). Only the amplitude of the neutral modes is shown in the figure because $\psi_n'^m$ is real (i.e., no latitudinal tilting). Most of the neutral modes have singularity at the critical latitude (indicated by +) where the angular speed of the neutral mode is equal to that of the zonal flow. Exceptionally the antisymmetric mode indicated by ▲ has no singularity. It is so-called "non-singular neutral mode". The latitudinal profile of the mode is $P_3^2(\mu)$, which corresponds to Rossby-Haurwitz wave with ($m = 2, n = 3$). The angular speed of this mode is $c_r = -\Omega/6$ in figure 2(d).

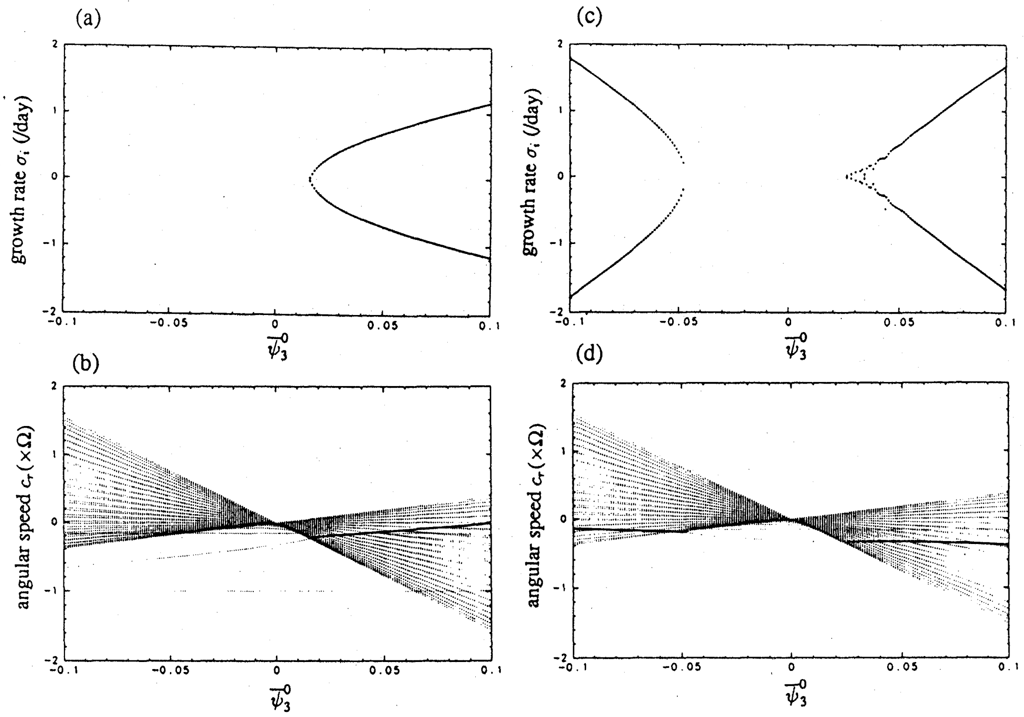

Figure 2. Eigenvalues for the stability of zonal flow $-1 \leq \overline{\psi}_3^0 \leq 1$. (a) growth rate of $m = 1$ eigenmodes, (b) angular speed of $m = 1$ modes, (c) growth rate of $m = 2$ modes, and (d) angular speed of $m = 2$ modes. Large dots correspond to $\sigma_i \neq 0$ eigenvalues.

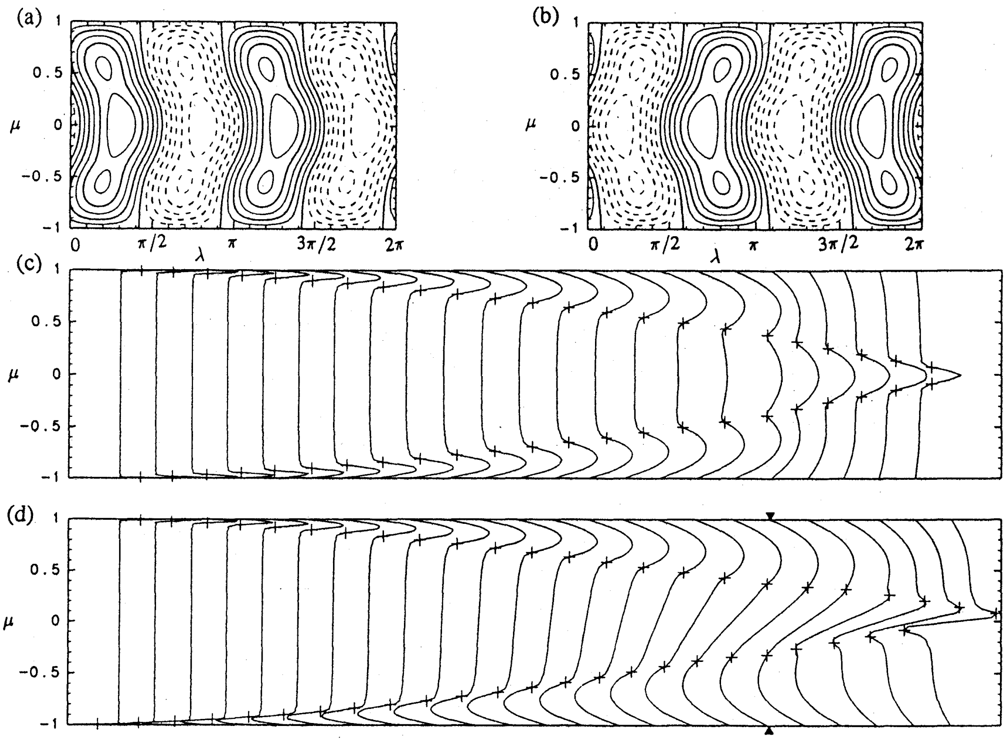

Figure 3. $m = 2$ eigenmodes for $\overline{\psi}_3^0 = -0.1$. (a) growing mode, (b) damping mode, (c) symmetrical neutral modes, and (d) antisymmetrical neutral modes. Symbol + in (c) and (d) indicates the critical latitude μ where $\overline{\omega}(\mu) = c$ for each mode. A non-singular neutral mode is indicated by ▲.

4. Nonlinear stability

If the zonal flow is linearly unstable, wave disturbances with small amplitude grow. However, the growth stops sometime due to the nonlinearity of the system (2.1). The spectral vorticity equation (2.6) with $N = 21$ is integrated numerically from various $\overline{\psi}_3^0$ with very small disturbances to investigate the nonlinear saturation of the disturbances. A spectral transform method is used in computing the nonlinear term. Figure 4 shows a couple of examples of the saturation for the initial state of $\overline{\psi}_3^0 = -0.06$ (a) and $\overline{\psi}_3^0 = 0.03$ (b): the normalized zonal enstrophy (solid line) and the wave enstrophy (dashed line) as a function of time. Initially the wave disturbance grows exponentially to attain a maximum at day 8.5 in figure 4(a), and at day 16.0 in (b). After the maximum, damping oscillation with small fluctuations is observed with a period of $\sim$ 5 days (a) and $\sim$ 12 days (b). The streamfunction field at the time when the wave enstrophy has a maximum is shown in figure 5. The wave of zonal wavenumber 2 is dominant in (a), and wavenumber 1 in (b). These results are consistent with the eigenvalue analysis in §3. Trough and ridge lines have no tilt with latitude at that time, which is indicative of little latitudinal vorticity flux.

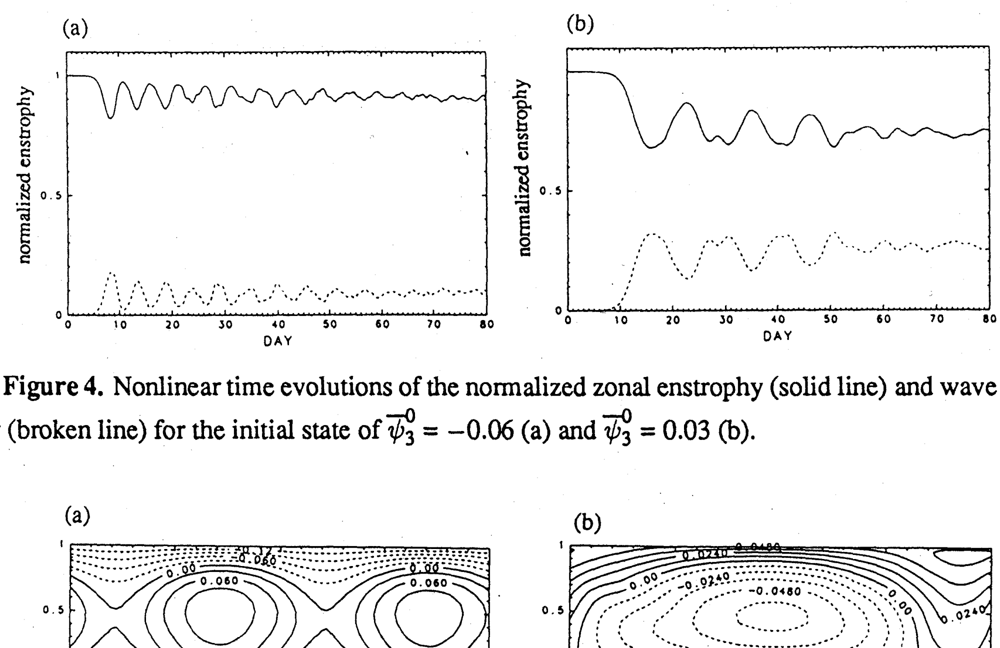

Figure 4. Nonlinear time evolutions of the normalized zonal enstrophy (solid line) and wave enstrophy (broken line) for the initial state of $\overline{\psi}_3^0 = -0.06$ (a) and $\overline{\psi}_3^0 = 0.03$ (b).

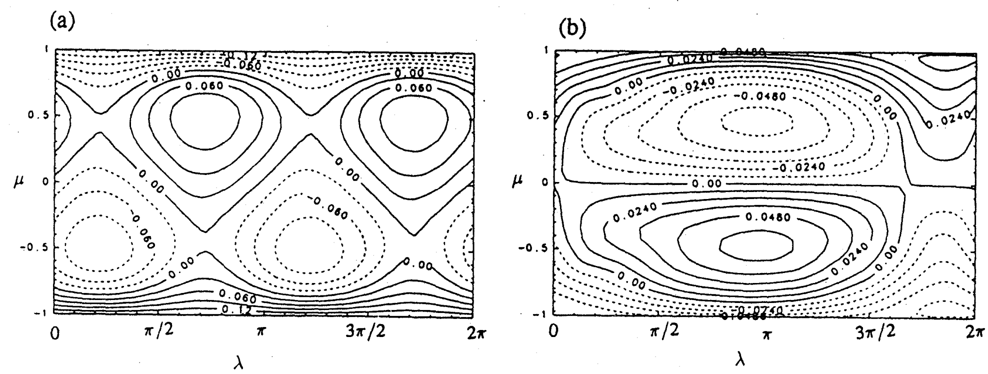

Figure 5. Streamfunction field at the time the wave enstrophy has a maximum value. (a) day 8.5 for the case of $\overline{\psi}_3^0 = -0.06$, (b) day 16.0 for the case of $\overline{\psi}_3^0 = 0.03$.

Now we consider the upper bounds of the growth of disturbances following Shepherd(1988). In the system (2.1), there are some invariants, one of which is the total enstrophy F as shown in figure 4.

$$\frac{d}{dt}F = 0, \qquad F \equiv \iint \frac{1}{2}\zeta^2 dS, \tag{4.1}$$

where

$$\zeta \equiv \nabla^2\psi, \qquad \iint dS \equiv \frac{1}{4\pi}\int_0^{2\pi} d\lambda \int_{-1}^{1} d\mu. \tag{4.2}$$

The total enstrophy F is divided into the zonal enstrophy F_z and the wave enstrophy F_w:

$$F = F_z + F_w = \iint \frac{1}{2}\bar{\zeta}^2 dS + \iint \frac{1}{2}\zeta'^2 dS, \tag{4.3}$$

where

$$\bar{\zeta} \equiv \int_0^{2\pi} \zeta d\lambda, \qquad \zeta' \equiv \zeta - \bar{\zeta}. \tag{4.4}$$

The wave enstrophy F_w is obviously bounded as

$$F_w \leq F_z + F_w = F. \tag{4.5}$$

Namely, the wave enstrophy at any time is less than the total enstrophy, or the zonal enstrophy at the initial state in the present case.

Shepherd(1988) obtained a more rigorous bound on F_w using the generalized Rayleigh theorem(McIntyre & Shepherd 1987):

$$F_w \leq \frac{|Q_\mu|_{\max}}{|Q_\mu|_{\min}} \iint \frac{1}{2}(q_0 - Q)^2 dS, \tag{4.6}$$

where $Q(\mu)$ is any monotonic continuous function of μ, and subscript 0 means its initial value. By utilizing equations (4.7) and (4.8) in Shepherd(1987)'s proof of the generalized Rayleigh theorem, we can obtain a more rigorous bound as follows;

$$F_w \leq |Q_\mu|_{\max} \iint |A(Q, q_0)| dS, \tag{4.7}$$

where $A(Q,q)$ is defined with $Y(Q)$, the inverse function of $Q(\mu)$:

$$A(Q,q) \equiv -\int_Q^q \{Y(\eta) - Y(Q)\} d\eta. \tag{4.8}$$

Note that the present notation of q is different from his definition; the present q is identical to his $Q + q$. Let the initial wave disturbance be infinitesimal, then (4.6) and (4.7) can be rewritten as;

$$F_w \leq \frac{|Q_\mu|_{\max}}{|Q_\mu|_{\min}} \frac{1}{2} \int_{-1}^{1} \frac{1}{2}(\bar{q}_0 - Q)^2 d\mu \equiv B_1(Q), \tag{4.9}$$

$$F_w \leq |Q_\mu|_{\max} \frac{1}{2} \int_{-1}^{1} |A(Q, \bar{q}_0)|\, d\mu \equiv B_2(Q). \tag{4.10}$$

Now we can obtain the bounds by calculating the minimum value of $B_1(Q)$ and $B_2(Q)$ for any $Q(\mu)$. If we denote the minima by $B_{1\min}$ and $B_{2\min}$, they satisfy the following inequality;

$$F_w \leq B_{2\min} \leq B_{1\min} \leq F. \tag{4.11}$$

We calculate $B_{1\min}$ and $B_{2\min}$ numerically with $\Delta\mu = 0.02$, employing a library subroutine for minimization by quasi-Newton methods. Figure 6 shows the normalized nonlinear bounds, $B_{1\min}/F$ (dashed line) and $B_{2\min}/F$ (solid line), for negative $\overline{\psi}_3^0$ (a) and positive $\overline{\psi}_3^0$ (b) together with the results of nonlinear time-integrations (open circles), which is the first maximum of the normalized wave enstrophy $(F_w/F)_{\max}$. The bounds of $B_{1\min}$ and $B_{2\min}$ are larger than $(F_w/F)_{\max}$ several times, but $B_{1\min}$ and $B_{2\min}$ well estimate the dependency of $(F_w/F)_{\max}$ on $\overline{\psi}_3^0$. As expected from (4.11), $B_{2\min}$ gives more rigorous and smooth bound than $B_{1\min}$ gives.

In computing $B_{2\min}$, we also obtain $Q(\mu)$ which gives the minimum value. In figure 7, the profile of $Q(\mu)$ is compared with $\overline{q}(\mu)$ at the time (F_w/F) takes the first maximum; the initial state $\overline{\psi}_3^0 = -0.06$ (a) and 0.03 (b). At this time $\overline{q}(\mu)$ satisfies Rayleigh's sufficient condition for stability. The profile $Q(\mu)$ gives a good estimate of $\overline{q}(\mu)$, particularly in the region where Q_μ is small.

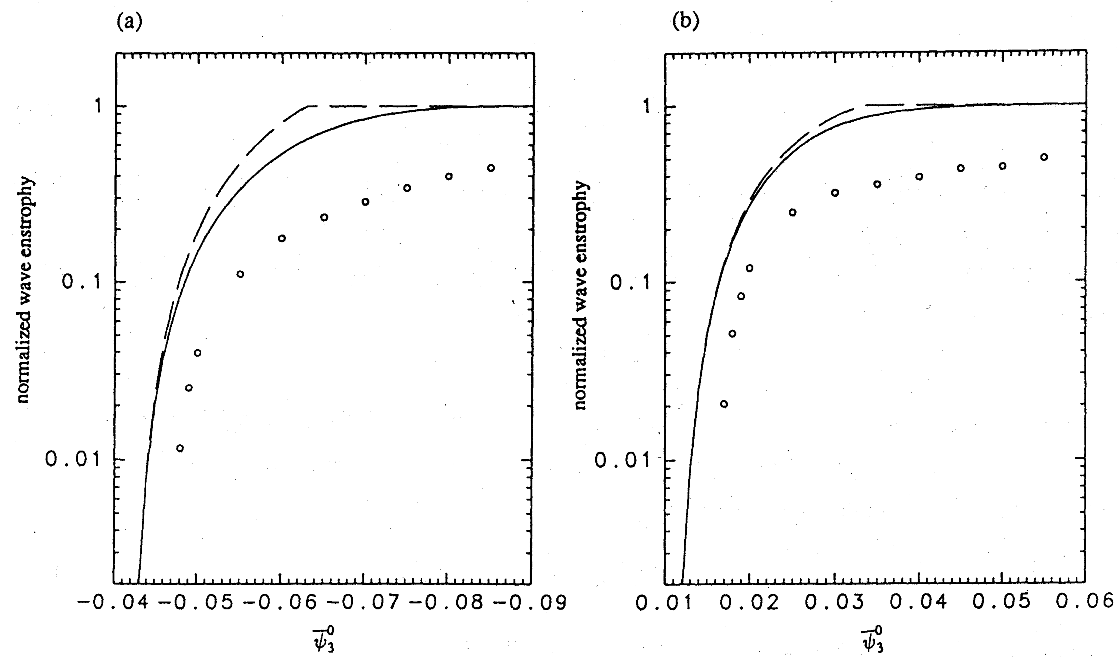

Figure 6. Bounds of the normalized wave enstrophy. The dashed curve shows Shepherd's original bound $B_{1\min}/F$, and the solid curve the present bound $B_{2\min}/F$. Open circles denote the first maximum of F_w/F obtained by nonlinear time integrations: (a) $\overline{\psi}_3^0 < 0$, (b) $\overline{\psi}_3^0 > 0$.

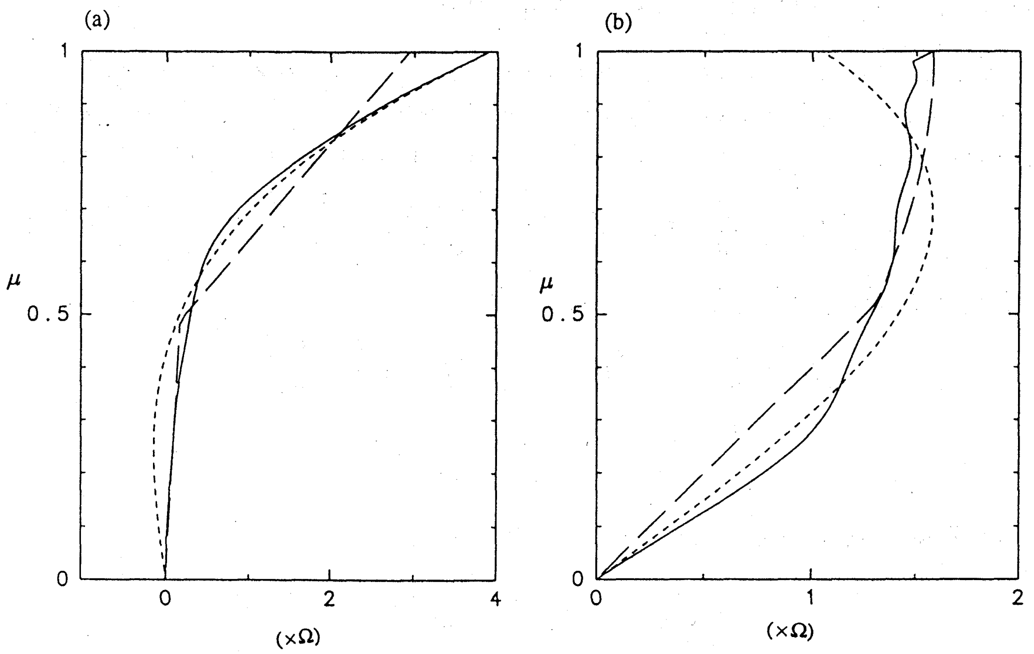

Figure 7. Latitudinal profile of absolute vorticity: the initial state $\bar{q}_0(\mu)$ (dotted line), $Q(\mu)$ which gives the present bound $B_{2\min}$ (dashed line), that obtained by nonlinear time-integrations (solid line). (a) $\overline{\psi}_3^0 = -0.06$, (b) $\overline{\psi}_3^0 = 0.03$.

5. Discussions

In this paper we have studied the linear and nonlinear stability of zonal flows largely based on the works by Baines(1976) and Shepherd(1988).

In the linear stability analysis of a zonal flow, we took advantage of remarkable advance in computational environments during these two decades. We showed the unstable mode is continuously connected to a Rossby-Haurwitz wave with a limit of the zonal flow to zero (figure 2). In addition, all of the eigenmodes are obtained as shown in figure 3: a stable and an unstable modes, singular neutral modes, and a nonsingular neutral mode for $m = 2$. The singular neutral modes come from the spectral form (2.13) with a truncation at a finite number N and correspond to the continuous mode in infinite dimension.

In the nonlinear stability analysis, the present bound is very useful to estimate the dependency on the zonal-flow amplitude (figure 6). In the process of nonlinear saturation, Rayleigh's instability criterion becomes unfulfilled due to the growing mode. Furthermore, the zonal absolute vorticity after the saturation is close to that obtained in the time integrations (figure 7).

We conclude that Shepherd's theorem is very useful to understand the nonlinear time evolution of the linearly unstable zonal flow and wave disturbances.

DEIG1 and DMINF1 of FUJITSU SSL2 Library provided by Data Processing Center, Kyoto University was used in §3 and §4, and GFD-DENNOU Library produced by M. Shiotani and S. Sakai was used for drawing the figures.

Appendix A Linear stability of zonal flow $\overline{\psi}_L^0$

THEOREM 1. *The zonal flow of $\overline{\psi}_L^0$ is stable to wave disturbances with wavenumber $m \geq L$.*

Proof. First, we derive time-differential equations on the energy and enstrophy of wave disturbances (Baines 1976). Multiplying (2.8) by ψ', integrating over the sphere and using Green's theorem yields

$$\frac{d}{dt}E_w \equiv -\frac{d}{dt}\frac{1}{2}\iint \psi'\nabla^2\psi' dS = -\frac{1}{a^2}\iint \psi'\frac{\partial\nabla^2\psi'}{\partial\lambda}\frac{d\overline{\psi}}{d\mu}dS, \tag{A.1}$$

and multiplying (2.8) by $\nabla^2\psi'$ yields

$$\frac{d}{dt}F_w \equiv \frac{d}{dt}\frac{1}{2}\iint (\nabla^2\psi')^2 dS = \frac{1}{a^2}\iint \psi'\frac{\partial\nabla^2\psi'}{\partial\lambda}\frac{d\nabla^2\overline{\psi}}{d\mu}dS. \tag{A.2}$$

Since

$$\nabla^2\overline{\psi} = -\frac{1}{a^2}N(N+1)\overline{\psi}, \tag{A.3}$$

eliminating $\overline{\psi}$ between (A.1) and (A.2) gives

$$\frac{d}{dt}F_w = \frac{1}{a^2}L(L+1)\frac{d}{dt}E_w. \tag{A.4}$$

If we consider a wave disturbance with zonal wavenumber m,

$$\psi' = a^2\Omega\cdot 2\sum_{n=m}^{N} Re(\psi_n'^m(t)e^{im\lambda})P_n^m(\mu), \tag{A.5}$$

then E_w and F_w can be written as

$$E_w = a^2\Omega^2\sum_{n=m}^{N} n(n+1)|\psi_n'^m|^2 = a^2\Omega^2\sum_{n=m}^{N} E_n^m(t), \tag{A.6}$$

$$F_w = \Omega^2\sum_{n=m}^{N}\{n(n+1)\}^2|\psi_n'^m|^2 = \Omega^2\sum_{n=m}^{N} n(n+1)E_n^m(t). \tag{A.7}$$

Substituting (A.6) and (A.7) into (A.4) yields

$$\sum_{n=m}^{N}\{n(n+1) - L(L+1)\}\frac{d}{dt}E_n^m(t) = 0. \tag{A.8}$$

Now, if the zonal flow is unstable to this disturbance, a growing eigenmode with growth rate $\sigma_i > 0$ must exist. For the growing mode, each $E_n^m(t)$ can be written as

$$E_n^m(t) = E_n^m(0)e^{2\sigma_i t}. \tag{A.9}$$

Substituting (A.9) into (A.8) and using $\sigma_i \neq 0$ yields

$$\sum_{n=m}^{N}\{n(n+1) - L(L+1)\}E_n^m(0) = 0. \tag{A.10}$$

If $m \geq L$, every $\{\cdots\}$ term in the summation is positive except for the case $n = m = L$. Then the growing mode must be consist of $\psi_L'^L$ mode only. However, this mode cannot grow nor damp solely, so the zonal flow $\overline{\psi}_L^0$ is stable to wave disturbances with zonal wavenumber $m \geq L$.

Q.E.D.

THEOREM 2. *The zonal flow of $\overline{\psi_2^0}$ is stable to any disturbances.* (Another proof is given by Baines(1976).)

Proof. From THEOREM 1, zonal flow of $\overline{\psi_2^0}$ can be unstable to disturbances with $m = 1$. In this case, any growing mode must contain $\psi_1'^1$ mode. (If not, from (A.10) the growing mode must be consist of $\psi_2'^1$ mode only and then cannot grow.) However, $\psi_1'^1$ mode must remain constant, because $D_{1n_1}^1 = 0$ in (2.11). This means the conservation of total angular momentum. Then such a growing mode cannot exist, so the zonal flow of $\overline{\psi_2^0}$ is stable to any disturbances.

Q.E.D.

Appendix B Integral theorems on a sphere

Here we summarize well-known classical integral theorems in the spherical domain. Further details on a plane are described in Kundu(1990) (on 1 and 2), and in Pedlosky(1964) (on 3).

1. Rayleigh's theorem (Baines 1976)

If the zonal flow is unstable, $\overline{q}_\mu$ must change its sign somewhere. (This is a necessary condition for instability.)

2. Fjϕrtoft's theorem

If the zonal flow is unstable, $(\overline{\omega} - \overline{\omega}_0)\overline{q}_\mu > 0$ somewhere for any $\overline{\omega}_0$. (This is a necessary condition for instability and includes Rayleigh's theorem.)

3. Semicircle theorem

If the zonal flow is unstable, the complex angular speed $c = c_r + ic_i$ for growing modes lie in the semicircle ;

$$\left\{\left(c_r - \frac{\overline{\omega}_{\max} - \overline{\omega}_{\min}}{2}\right)^2 + c_i^2\right\} \leq \left(\frac{\overline{\omega}_{\max} - \overline{\omega}_{\min}}{2}\right)^2 + \frac{|\Omega + \overline{\omega}|_{\max}}{m^2 + 1 + \dfrac{m^2}{2|m| + 3}}, \qquad (B.1)$$

$$c_i > 0.$$

And if $\Omega + \overline{\omega}_{\min} \geq 0$, then $c_r \leq \overline{\omega}_{\max}$.

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